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Localization and Ideal Theory in Iterated Differential Operator Rings

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In this paper we study primality, hypercentrality, simplicity, and localization and the second layer condition in skew enveloping algebras and iterated differential operator rings. We give sufficient conditions for the skew enveloping algebra of a nilpotent Lie algebra with coefficient ring containing the rational numbers to be a simple ring, and we give necessary and sufficient conditions in the case that the Lie algebra is Abelian. Our main results show that if L is a finite dimensional solvable Lie algebra over a field k of characteristic zero and R is an Artinian or a commutative Noetherian algebra over k, then the skew enveloping algebra $R\# U(L)$ satisfies the second layer condition. We discuss consequences of this for localization and use the localization theory to state a classical Krull dimension versus global dimension inequality when k is uncountable.

1. INTRODUCTION AND DEFINITIONS

In recent years there has been a great deal of study of the universal enveloping algebras of finite dimensional Lie algebras [3, 11]. If we introduce a coefficient ring and allow the Lie algebra to act on the coefficient ring, we may simultaneously study another interesting class of Noetherian rings, the class of differential operator rings. In this paper we take this tack and study skew enveloping algebras, deriving results analogous to those we derived for skew group rings and group-graded rings in a previous paper [1]. In particular, we study primality, hypercentrality, simplicity, and especially the second layer condition in skew enveloping algebras.

In the study of both enveloping algebras and differential operator rings, localization has been a valuable tool. Some of the main results in this paper involve questions of localization in skew enveloping algebras. Recently a theory of localization in Noetherian rings has been developed (see Sect. 7 as well as [22] and [7]) which applies to enveloping algebras of finite

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dimensional solvable Lie algebras. Most of Section 7 is devoted to proving that if L is a nilpotent Lie algebra over a field of positive characteristic or a solvable Lie algebra over a field k of characteristic 0, then the skew enveloping algebra $R \# U(L)$ satisfies Jategaonkar's second layer condition (which must hold for the localization theory to apply) for a large class of Noetherian k-algebras R, including commutative Noetherian and Artinian k-algebras. In Proposition 7.6, we use this result and a result of Sigurdsson's, along with the general localization theory, to show that if k is uncountable and has characteristic 0 and R is a Noetherian p.i. algebra, then all prime ideals in $R \# U(L)$ belong to cliques which are localizable. We then apply the localization results to show that the classical Krull dimension of $R \# U(L)$ is bounded by its global dimension in this situation. We also give some conditions guaranteeing that certain single prime ideals are localizable.

The methods of proof are ring-theoretic, involving the use of iterated differential operator rings. The only Lie theory we use is a skew version of the Poincaré–Birkhoff–Witt Theorem which enables us to characterize $R \# U(L)$ and use degree arguments.

Our results in Section 7 depend on the use of the Artin–Rees property and the existence of "enough" central elements. In Proposition 6.2 we show that the skew enveloping algebra $R \# U(L)$ is a hypercentral ring when L is nilpotent and R is L-hypercentral (see Sect. 6 for the definition). We use this result to give conditions for $R \# U(L)$ to be simple if k is a field of characteristic zero. For example, in Theorem 6.3 we show that if L is nilpotent with basis $x_1, ..., x_t$ and R is L-simple, then $R \# U(L)$ is simple if the derivations corresponding to the x_i are linearly independent, modulo certain inner derivations, over the subfield of central $Z(L)$-constants of R. If L is Abelian it is easily shown that these conditions are necessary and sufficient.

The first half of the paper is devoted to general properties of skew enveloping algebras and sets of derivations acting on rings, some of which we need later. In Section 2 we state some basic results about a set Δ of derivations acting on a ring R. We give several characterizations of Δ-prime and Δ-semiprime ideals. Most of these results are known in special cases.

In Section 3 we define skew universal enveloping algebras and discuss some of their elementary properties. We also discuss the nature of $R \# U(L)$ as an iterated differential operator ring; we return to this discussion in Section 6. In Section 4 we discuss degree and primality in skew enveloping algebras, and we determine the prime radical of a skew enveloping algebra.

In Section 5 we derive some results on the Goldie and Jacobson conditions. In Proposition 5.2 we show that for Lie algebras which are free modules and rings R with no $\mathbb{Z}$-torsion, R is semiprime right Goldie if and only if the skew enveloping algebra $R\# U(L)$ is semiprime right Goldie. Proposition 5.3 states a consequence of a theorem of Jordan: it states that if R is a right Noetherian Jacobson ring and L is completely solvable, then $R\# U(L)$ is a Jacobson ring.

All rings and algebras in this paper, except Lie algebras, will be associative and have an identity. In particular, throughout this paper k will be a commutative ring with 1, R an associative k-algebra with 1, Δ a set of k-linear derivations on R, and L a Lie k-algebra which is finitely generated as a k-module. We will assume that L acts on R via a Lie algebra map δ from L to the Lie algebra $\mathrm{Der}_k(R)$ of k-linear derivations of R. (If δ_1 and δ_2 are derivations of R, then $[\delta_1, \delta_2]=\delta_1\delta_2-\delta_2\delta_1$ is also a derivation on R.) For $x\in L$, we will denote the derivation $\delta(x)$ by δ_x. The most important cases are where k is a field or $k=\mathbb{Z}$ and L is a free k-module.

The notation $[x, y]$ will have two meanings. If x and y are elements of a Lie algebra L, then $[x, y]$ will be the usual Lie product, while if they are elements of an associative ring R, then $[x, y]$ will be the additive commutator $xy-yx$.

We will use $\mathbb{N}$ to denote the set of nonnegative integers, $\mathbb{Z}$ to denote the ring of integers, and $\mathbb{Q}$ to denote the field of rational numbers. We will use $\subset$ to indicate strict inclusion. An adjective like "Noetherian," when applied to a ring, is understood to apply on both sides; hence a Noetherian ring is the same as a right and left Noetherian ring. Similarly, an ideal is said to have the AR property if it has the right and left AR properties, and so on.

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2. Δ-Prime and Δ-Semiprime Ideals

Before we study the ideal theory of skew enveloping algebras, we prove some general results about sets of derivations of a k-algebra R. Let Δ be a set of k-linear derivations on R (e.g., $\Delta=\{\delta_x \mid x\in L\}$ for a Lie algebra L acting on R via a map δ). We say a right ideal I of R is *Δ-invariant* if $\delta(I)\subseteq I$ for each $\delta\in\Delta$. An ideal I of R is said to be *Δ-maximal* if it is Δ-invariant and the only Δ-invariant ideal of R properly containing I is R itself; the ring R is said to be *Δ-simple* if 0 is a Δ-maximal ideal of R. An ideal I of R is *Δ-prime* if it is Δ-invariant and for any Δ-invariant ideals J_1, J_2 of R with $J_1J_2\subseteq I$, either $J_1\subseteq I$ or $J_2\subseteq I$; we define *Δ-semiprime* in the same way with $J_1=J_2$. We say R is *Δ-prime* [resp. *Δ-semprime*] if 0 is a Δ-prime [resp. Δ-semiprime] ideal of R. Clearly the ideal I is Δ-prime if and only if I is Δ-invariant and R/I is Δ-prime.

In contrast to the case of group actions, the prime radical and Jacobson radical need not be Δ-invariant. For example, let k be a field of characteristic 2, let $R = k[t]/(t^2)$, and let $\delta(\overline{p(t)}) = \overline{p'(t)}$: here $J(R) = (\bar{t})$ is nilpotent, yet $\delta(J(R)) = \delta(R) = k$. (This example is essentially the same as [23, Example 2.3].) However, as observed in [3, Lemma 4.1, p. 34], the center of R is always invariant under each $\delta \in \Delta$. To see this, suppose $r \in R$ and $z \in \operatorname{cen} R$. Then

$$\delta(z)\, r - r\delta(z) = (\delta(z)\, r + z\delta(r)) - (r\delta(z) + \delta(r)\, z) = \delta(zr) - \delta(rz) = 0,$$

so $\delta(z) \in \operatorname{cen} R$.

Many of the results in this section were stated by Jordan in [23] and by Goodearl and Warfield in [16] for the case where Δ consists of a single derivation. Many of the proofs we give are straightforward modifications of the original proofs for this special case, but we include some of them for the sake of completeness and because of the occasional differences.

The next results give alternate characterizations of Δ-primality and Δ-semiprimality.

PROPOSITION 2.1. *Let I be a Δ-invariant ideal of R. Then the following conditions are equivalent*:

(a) *I is Δ-prime.*

(b) *If J_1 and J_2 are right ideals of R, one of which is Δ-invariant, with $J_1 J_2 \subseteq I$, then either $J_1 \subseteq I$ or $J_2 \subseteq I$.*

(c) *If $a, b \in R$ and $\delta_1 \cdots \delta_l(a)\, R\delta'_1 \cdots \delta'_m(b) \subseteq I$ for any $\delta_1, \dots, \delta_l, \delta'_1, \dots, \delta'_m \in \Delta$ (possibly $l = 0$ or $m = 0$), then either $a \in I$ or $b \in I$.*

(d) *If $a, b \in R$ and $\delta_1 \cdots \delta_l(a)\; Rb \subseteq I$ for any $\delta_1, \dots, \delta_l \in \Delta$ (possibly $l = 0$), then either $a \in I$ or $b \in I$.*

Proof. The implications (b) → (a) and (d) → (c) are trivial. The reverse implications (a) → (b) and (c) → (d) are fairly straigthforward to prove using the properties of derivations. For example, to prove (a) → (b), one assumes J_1 and J_2 are as in (b), with say J_1 Δ-invariant, and first notes that one may replace J_k with RJ_k and hence assume J_1 and J_2 are ideals. One next shows that if J'_2 is the Δ-invariant ideal generated by J_2, then $J_1 J'_2 \subseteq I$ and so (a) can be applied.

Thus we need only show the equivalence of (a) and (c). This again is straigthforward. For example, given Δ-invariant ideals J_1 and J_2 of R with neither one contained in I but $J_1 J_2 \subseteq I$, pick $a \in J_1 \backslash I$ and $b \in J_2 \backslash I$. The elements a, b violate condition (c).

Conversely, given a, b as in (c), if J_1 is the Δ-invariant ideal of R generated by a and J_2 is the Δ-invariant ideal generated by b, then $J_1 J_2 \subseteq I$ and hence by (a) either $a \in J_1 \subseteq I$ or $b \in J_2 \subseteq I$.

PROPOSITION 2.2. *If I is a Δ-invariant ideal of R, then I is Δ-semiprime if and only if for any $a \in A$ such that $\delta_1 \cdots \delta_l(a)\ Ra \subseteq I$ for all $\delta_1, \ldots, \delta_l \in \Delta$ (possibly $l = 0$), we have $a \in I$.*

The proof of Proposition 2.2 is similar to that of Proposition 2.1. Of course we could restate all four conditions of Proposition 2.1 in suitably altered form in Proposition 2.2. Note that all conditions can be switched to their left-right duals.

PROPOSITION 2.3. *An ideal of R is Δ-semiprime if and only if it is an intersection of Δ-prime ideals.*

Proof. Sufficiency is obvious. To prove necessity, it suffices to show that if R is a Δ-semiprime ring and a is a nonzero element of R, then there is a Δ-prime ideal P of R such that $a \notin P$. To prove this we imitate the standard proof that a semiprime ideal is an intersection of prime ideals.

Suppose then that R is Δ-semiprime and $a \in R \backslash \{0\}$. Set $a_1 = a$. We can define a sequence of nonzero elements $a_1, a_2, \ldots,$ of R such that each $a_{n+1} = \delta_{1,n} \cdots \delta_{k(n),n}(a_n)\ b_n a_n$ for some $b_n \in R$ and $\delta_{1,n}, \ldots, \delta_{k(n),n} \in \Delta$. In fact, given a_n, the existence of such a nonzero a_{n+1} is immediate from Δ-semiprimality and Proposition 2.2. Let P be a Δ-invariant ideal of R maximal with respect to not containing any a_n. By definition, $a \notin P$; we will now show P is Δ-prime.

Suppose I and J are Δ-invariant ideals properly containing P. Then $a_i \in I$ and $a_j \in J$ for some i and j. If n is the maximum of i and j, then clearly $a_n \in I \cap J$. Since I is Δ-invariant, this implies that $a_{n+1} = \delta_{1,n} \cdots \delta_{k(n),n}(a_n)\ b_n a_n \in IJ$. Thus IJ is not contained in P. This shows P is Δ-prime, and completes the proof of the proposition.

Given a right ideal I of R, we need to introduce a related Δ-invariant right ideal of R. We denote by $(I : \Delta)$ the largest Δ-invariant right ideal of R contained in I. Using induction and properties of derivations, one can check that

$$(I : \Delta) = \{ i \in I \mid \delta_1 \cdots \delta_n(i) \in I \quad \text{for all } \delta_1, \ldots, \delta_n \in \Delta \}.$$

Furthermore, if I is an ideal of R, then $(I : \Delta)$ is an ideal of R.

We now prove a series of results which enable us to relate prime ideals and Δ-prime ideals of R.

LEMMA 2.4. *Let I be a prime [resp. semiprime] ideal of R. Then $(I : \Delta)$ is a Δ-prime [resp. Δ-semiprime] ideal of R.*

Proof. Suppose J_1 and J_2 are Δ-invariant ideals of R and $J_1 J_2 \subseteq (I : \Delta) \subseteq I$. Then if I is prime, either $J_1 \subseteq I$ of $J_2 \subseteq I$. Thus by Δ-invariance, either $J_1 \subseteq (I : \Delta)$ or $J_2 \subseteq (I : \Delta)$. If I is semiprime and $J_1 = J_2$, we get $J_1 \subseteq (I : \Delta)$.

PROPOSITION 2.5. *Let I be a semiprime ideal of R such that R/I has no $\mathbb{Z}$-torsion.*

(a) *The ideal $(I : \Delta)$ is a semiprime ideal of R.*

(b) *If I is prime, then $(I : \Delta)$ is prime. If I is a minimal prime ideal of R, then I is a Δ-invariant ideal.*

Proof. For the case where I is prime, see [11, Lemma 3.3.2, p. 107]. When I is semiprime, the same proof works with appropriate modification.

The following result is standard; a similar result appears in [16, Sect. 1].

LEMMA 2.6. *If R is a Δ-prime ring, then either R has no $\mathbb{Z}$-torsion or there is a prime $p \in \mathbb{Z}$ such that $pR = 0$.*

The following result is an appropriately modified version of [16, Lemma 1.2, Proposition 1.3; 23, Lemma 2.1]. We omit the proof.

PROPOSITION 2.7. *Let N be the prime radical of R and suppose that either R or R/N has no $\mathbb{Z}$-torsion. If I is an intersection of minimal prime ideals of R, then R/I has no $\mathbb{Z}$-torsion and I is a Δ-invariant ideal. In particular, N is a Δ-semiprime ideal of R.*

The following result is a generalization of a part of [23, Theorem 2.2] and the proof is essentially the same.

THEOREM 2.8. *If R has the ascending chain condition* (a.c.c.) *on left and right annihilator ideals and R is Δ-prime, then the prime radical N of R is a nilpotent prime ideal and $(N : \Delta) = 0$. If in addition R has no $\mathbb{Z}$-torsion, then R is prime.*

Proof. Let P be a maximal proper left annihilator ideal of R. One can show that $(P : \Delta) = 0$ and that P is a nilpotent prime ideal of R as in the proof of [23, Theorem 2.2, (ii) $\rightarrow$ (iii)], so $P = N$.

If R has no $\mathbb{Z}$-torsion, then Proposition 2.7 implies that P is a nilpotent Δ-invariant ideal of R, and hence $P = 0$.

COROLLARY 2.9. *Let R have the* a.c.c. *on ideals, I be a Δ-semiprime ideal of R, and N be the prime radical of I in R.*

(a) *Then $(N : \Delta) = I$ and $N^m \subseteq I$ for some positive integer m; if I is Δ-prime, then N is prime.*

(b) *Suppose R/I has no $\mathbb{Z}$-torsion. Then I is semiprime and if in addition I is Δ-prime, then I is prime.*

Proof. The a.c.c. on ideals implies that $N^m \subseteq I$ for some m, so certainly $(N : \Delta)^m \subseteq I$. By Δ-semiprimality, this means that $(N : \Delta) \subseteq I$. Since $I \subseteq N$, we must have $I = (N : \Delta)$. The first follows from the previous result.

In [16, Example 1.6], Goodearl and Warfield give an example showing that the last result fails to hold without appropriate chain condition on R. Note also that the example given in the second paragraph of this section shows that a commutative Artinian Δ-simple ring need not be prime or even semiprime.

COROLLARY 2.10. *If $R \supseteq \mathbb{Q}$ and R has the* a.c.c. *on ideals, then an ideal of R is Δ-semiprime if and only if it is a Δ-invariant semiprime ideal and an ideal of R is Δ-prime if and only if it is a Δ-invariant prime ideal of R.*

LEMMA 2.11. *Let R have the* a.c.c. *on ideals.*

(a) *A Δ-prime ideal I of R is a minimal Δ-prime ideal of R if and only if the prime radical P of I is a minimal prime ideal of R.*

(b) *A prime ideal P of R is a minimal prime ideal of R if and only if $(P : \Delta)$ is a minimal Δ-prime ideal of R.*

Proof. (a) $(\rightarrow)$ If Q is a prime ideal and $Q \subseteq P$, then $(Q : \Delta)$ is a Δ-prime ideal by Lemma 2.4 and $(Q : \Delta) \subseteq (P : \Delta) = I$, so by minimality, $(Q : \Delta) = I$. This means $Q \supseteq I$, so since P is the prime radical of I, we see $Q = P$ and hence P is minimal.

$(\leftarrow)$ If J is a Δ-prime ideal contained in I, then the prime radical of J is contained in P and hence by minimality, the prime radical of J is P. Thus by Corollary 2.9, we have $J = (P : \Delta) = I$.

(b) This follows from (a) by Corollary 2.9.

We will need the next result later. It is due to Fisher [12, Theorems 1 and 2].

PROPOSITION 2.12. *If R is a right Noetherian Δ-semiprime ring, then R has a right Artinian right quotient ring Q. If R is Δ-prime and $\tilde{\Delta}$ is the set of derivations on Q induced by Δ, then $Q/J(Q)$ is simple and Q is a $\tilde{\Delta}$-simple ring.*

3. DEFINITION AND BASIC PROPERTIES OF SKEW ENVELOPING ALGEBRAS

Suppose the Lie k-algebra L acts on the k-algebra R via a Lie algebra map $\delta: L \rightarrow \mathrm{Der}_k(R)$. A *skew universal enveloping algebra* $R \# U(L)$ is an associative k-algebra U together with a k-algebra map $\psi_0: R \rightarrow U$ and a Lie

k-algebra map $\chi_0: L \to U$ (where U is regarded as a Lie algebra with commutator as Lie product) such that

$$\psi_0(r)\,\chi_0(x) - \chi_0(x)\,\psi_0(r) = \psi_0(\delta_x(r)) \qquad \text{for all } x \in L \quad \text{and} \quad r \in R, \quad (*)$$

which has the following universal property: for any associative k-algebra S and any k-algebra map $\psi: R \to S$ and any Lie k-algebra map $\chi: L \to S$ such that $(*)$ is satisfied when ψ replaces ψ_0 and χ replaces χ_0, there is a unique k-algebra map $\xi: U \to S$ such that $\xi\chi_0 = \chi$ and $\xi\psi_0 = \psi$. It is clear from the universal property that U is unique up to isomorphism if it exists and that it is generated as a k-algebra by $\psi_0(R) \cup \chi_0(L)$. (Skew enveloping algebras are discussed in [3, Sect. 4, pp. 44–47; 10, Sect. 2; 25, Sect. 2].)

In fact we can construct U as a free k-algebra on generators $\{\bar{r}\}_{r \in R} \cup \{\bar{x}\}_{x \in L}$ subject to the relations (i) $\bar{1}_L = \bar{1}_R$, (ii) $\bar{r} \cdot \bar{s} = \overline{rs}$, (iii) $\overline{(\sum_{i=1}^n \lambda_i r_i)} = \sum_{i=1}^n \lambda_i \bar{r}_i$, (iv) $\bar{x} \cdot \bar{y} - \bar{y} \cdot \bar{x} = \overline{[x, y]}$, (v) $\overline{(\sum_{i=1}^n \lambda_i x_i)} = \sum_{i=1}^n \lambda_i \bar{x}_i$, and (vi) $\bar{r} \cdot \bar{x} - \bar{x} \cdot \bar{r} = \overline{\delta_x(r)}$, where r, s, $r_1, \ldots, r_n \in R$ and x, y, $x_1, \ldots, x_n \in L$ and $\lambda_1, \ldots, \lambda_n \in k$. Together with the maps ψ_0 taking r to $\bar{r}$ and χ_0 taking x to $\bar{x}$, it is easy to check that U satisfies the required universal property. We can of course also define U in terms of a generating set for R as a k-algebra and a generating set for L as a k-module. Our remarks above show that if $\{x_1, \ldots, x_t\}$ generates L as a k-module, then U is generated as an algebra by $\psi_0(R)$ and $\chi_0(x_1), \ldots, \chi_0(x_t)$.

Using the relations $\bar{r} \cdot \bar{x} - \bar{x} \cdot \bar{r} = \overline{\delta_x(r)}$ and $\bar{x} \cdot \bar{y} - \bar{y} \cdot \bar{x} = \overline{[x, y]}$, one can show that U is generated as a right and left R-module by

$$\{\chi_0(x_t)^{n_t} \cdots \chi_0(x_1)^{n_1} \mid (n_t, \ldots, n_1) \in \mathbb{N}^t\}.$$

As usual in the theory of enveloping algebras, this enables us to define a filtration of U with nth part generated as a k-module by products $\chi_0(x_t)^{n_t} \cdots \chi_0(x_1)^{n_1}\,\psi_0(r)$, where $\sum_{i=1}^t n_i \leqslant n$. The graded algebra associated to this filtration is a homomorphic image of the polynomial algebra $R[y_1, \ldots, y_t]$. This implies that if R has the a.c.c. on right ideals or left ideals or two-sided ideals, then U has the same a.c.c. (because we are assuming L is finitely generated as a k-module). (For more information regarding this filtration and the other topics discussed above and below, see [19, Chap. V, Sects. 1–3].)

We will mainly be interested in the case where L is a free k-module with basis $\{x_1, \ldots, x_t\}$. In this case a skew version of the Poincaré–Birkhoff–Witt (PBW) Theorem states that both ψ_0 and χ_0 are injective mappings, so we may simply suppose L and R are subsets of U, and states that the *standard monomials* $\{x_t^{n_t} \cdots x_1^{n_1} \mid (n_t, \ldots, n_1) \in \mathbb{N}^t\}$ form a basis for U as a free right and left R-module (so as a right R-module, $R \# U(L)$ is isomorphic to $U(L) \otimes_k R$ where $U(L)$ is the ordinary enveloping algebra of L). If we take the filtration mentioned above and form the associated graded algebra, we get precisely the polynomial algebra $R[x_1, \ldots, x_t]$. Thus if R is a domain, U is a domain as in [19, Theorem V.4, pp. 164–165]. For k a field, this Skew PBW Theorem is proved in [3, Theorem 4.2, p. 36]. The general case can be proved using the standard techniques.

We note that the skew enveloping algebra $R \# U(L)$ can be regarded as the smash product of the $U(L)$-module algebra R and the Hopf algebra $U(L)$, as the notation indicates. This point of view is taken in [2].

Suppose L is a Lie k-algebra, N is an ideal of L, and H is a subalgebra of L such that $L = H \oplus N$, and H and N are free as k-modules. Then as with crossed products and group actions, one can define a natural action of H on $R \# U(N)$ such that $(R \# U(N)) \# U(H) \cong R \# U(L)$. This is the special case of Lie algebra extensions where L is a semidirect product of N by H.

If L is an Abelian Lie algebra which has basis $\{x_1, \ldots, x_t\}$ as a k-module, then the skew enveloping algebra $R \# U(L)$ is just the *multiple differential operator ring* $R[x_1, \ldots, x_t; \delta_1, \ldots, \delta_t]$, where the variables $x_1, \ldots, x_t$ commute and the derivations $\delta_i = \delta_{x_i}$ commute. We will also be interested in a more general kind of Lie algebra and a more general kind of differential operator ring. Let R be a ring. We define a sequence of ordinary differential operator rings $R_0, R_1, \ldots, R_t$ as follows. Set $R = R_0$ and for each $i > 0$, let δ_i be a derivation on R_{i-1} and let $R_i = R_{i-1}[x_i; \delta_i]$. We call the ring R_t an *iterated differential operator ring over R*. We call a Lie k-algebra L *completely solvable* if L is a free k-module with basis $\{x_1, \ldots, x_t\}$ and for any j, m with $1 \leqslant j < m \leqslant t$, we have $[x_j, x_m] = \sum_{i=1}^{j} \lambda_i x_i$ for some coefficients $\lambda_i \in k$. Using the remark in the last paragraph, it is clear that for such an L, the enveloping algebra $R \# U(L)$ is isomorphic to an iterated differential operator ring R_t as above, with the added condition that if $t \geqslant m > j \geqslant 0$, the ring R_j is stable under the derivation δ_m. (Not all iterated differential operator rings of this type are skew enveloping algebras over R.)

If k is a field and L is a finite dimensional solvable Lie k-algebra, then using the definition of solvability, one can choose a basis $x_1, \ldots, x_t$ of L such that $[x_j, x_m]$ is a linear combination of elements x_i where i ranges from 1 up to the maximum of $j-1$ and $m-1$. This implies that the ring $R \# U(L)$ is an iterated differential operator ring over R as described above. (In this case one does not have the same stability under succeeding derivations as in the completely solvable case.)

4. Ideal Theory in Skew Enveloping Algebras

We now turn to the study of skew enveloping algebras. If L is a Lie k-algebra acting as derivations on R, then we define *L-invariant ideals*, *L-prime ideals*, etc., in the obvious way, namely Δ-invariant ideals, Δ-prime ideals, and so on, where $\Delta = \{\delta_x \mid x \in L\}$. Note that if Δ is a set of derivations on R, then Δ generates a Lie subalgebra L of the Lie algebra $\mathrm{Der}_k R$ and the classes of Δ-invariant ideals and L-invariant ideals coincide. Thus the various concepts studied above are essentially equivalent, whether stated in terms of Δ or of L.

We note that if I is an L-invariant ideal of R and $S = R \# U(L)$, then $IS = SI$. As in [19, Theorem V.1(4), p. 153], one can show that $S/IS \cong (R/I) \# U(L)$, where we define the action of L on R/I in the natural way. If L is free as a k-module, then IS is the set of linear combinations of standard monomials with coefficients in I. Conversely, if J is an ideal of S, then $J \cap R$ is an L-invariant ideal of R with $(J \cap R) S \subseteq J$.

We now wish to give characterizations of primality and simplicity in some skew enveloping algebras. To do this, we need a notion of degree. Although we are restricting our attention to Lie algebras with a finite basis, our results can be generalized to those with an infinite basis.

We will make use of the following ordering on t-tuples of nonnegative integers, denoted $\leqslant_1$. If $\mathbf{n} = (n_t, \ldots, n_1) \in \mathbb{N}^t$, we denote $n_1 + \cdots + n_t$ by $|\mathbf{n}|$. If $\mathbf{m} = (m_t, \ldots, m_1) \in \mathbb{N}^t$, we say $\mathbf{n} <_1 \mathbf{m}$ if either $|\mathbf{n}| < |\mathbf{m}|$ or $|\mathbf{n}| = |\mathbf{m}|$ and at the largest i for which $n_i \neq m_i$ the relation $n_i < m_i$ holds. Clearly $\leqslant_1$ is a total order. This ordering is discussed in [11, Sect. 2.6, pp. 85, 86], where it is proved that $\leqslant_1$ induces a well-ordering on $\mathbb{N}^t$ isomorphic to the usual ordering on $\mathbb{N}$.

Fix a k-basis $x_1, \ldots, x_t$ of L: we define the *degree* of a nonzero monomial (in standard form) $x_t^{n_t} \cdots x_1^{n_1} r$ (where the coefficient r is in R) to be $\mathbf{n} = (n_t, \ldots, n_1) \in \mathbb{N}^t$. The *degree* of an arbitrary nonzero element $f = f_1 + \cdots + f_l$, where the elements f_i are distinct nonzero monomials, is the maximum degree of the monomials f_i with respect to the ordering $\leqslant_1$, and is denoted $\deg f$. (We set $\deg 0 = (-\infty, \ldots, -\infty)$.) The *leading term* of f is the monomial of highest degree and the *leading coefficient* is the coefficient of that monomial; if it is 1, we say f is *monic*.

It is clear that

$$\deg(f+g) \leqslant_1 \max(\deg f, \deg g)$$

and that

$$\deg(fg) \leqslant_1 \deg f + \deg g,$$

with equality in the latter inequality if either f or g is monic. In fact, if the product of the leading coefficients of f and g is non-zero, then clearly this product is the leading coefficient of fg and the degree of the product is the sum of the individual degrees.

If J is an ideal of $R \# U(L)$, let $\mathrm{ld}(J)$ be the set of leading coefficients of elements of J, including 0. Our last remark indicates that $\mathrm{ld}(J)$ is closed under right and left multiplication by elements of R. Multiplication on the left by appropriate monomials enables us to show any two nonzero elements of J have multiples of the same degree with the same leading coefficients as originally. Thus $\mathrm{ld}(J)$ is closed under addition and so is an ideal of R. In some cases more than this is true.

LEMMA 4.1. *If L is completely solvable and J is an ideal of $R \# U(L)$, then the set $\mathrm{ld}(J)$ of leading coefficients of elements of J is an L-invariant ideal of R.*

Proof. We saw above that $\mathrm{ld}(J)$ is an ideal. The proof of the lemma is based on the fact that if $f \in R \# U(L)$ and $l \in L$, then $\deg[f, l] \leqslant_1 \deg f$.

Let us temporarily assume this fact and let $f \in R \# U(L)$, say $f = f_1 r_1 + \cdots + f_m r_m$ where $r_1, \ldots, r_m \in R$ are nonzero and $f_1, \ldots, f_m$ are standard monomials with $\deg f_1 >_1 \cdots >_1 \deg f_m$. Thus r_1 is the leading coefficient of f and so $r_1 \in \mathrm{ld}(J)$.

If $l \in L$, then it is easy to check that $[f, l] = \sum_{i=1}^{m} ([f_i, l]\, r_i + f_i\, \delta_l(r_i))$. If $\delta_l(r_1) = 0$, then certainly $\delta_l(r_1) \in \mathrm{ld}(J)$. Suppose $\delta_l(r_1) \neq 0$, so that the degree of $\sum_{i=1}^{m} f_i\, \delta_l(r_i)$ is $\deg f_1$. By our assumption, $\deg[f_i, l] <_1 \deg f_1$ for any $i > 1$, so only the term $[f_1, l]\, r_1 + f_1\, \delta_l(r_1)$ can contain monomials of degree $\deg f_1$. Thus if r is the leading coefficient of $[f_1, l]$, we see that either $rr_1 + \delta_l(r_1) = 0$ or one of $\delta_l(r_1)$ or $rr_1 + \delta_l(r_1)$ is the leading coefficient of $[f, l]$. In any case, then, either $rr_1 + \delta_l(r_1) \in \mathrm{ld}(J)$ or $\delta_l(r_1) \in \mathrm{ld}(J)$, so since $r_1 \in \mathrm{ld}(J)$, we have $\delta_l(r_1) \in \mathrm{ld}(J)$.

Thus we have only to prove that $\deg[f, l] \leqslant_1 \deg f$ for any $f \in R \# U(L)$ and $l \in L$. It is clearly enough to prove this for a monomial f and $l = x_j$ for $1 \leqslant j \leqslant n$. Since $[fr, l] = [f, l]\, r + f\delta_l(r)$, it is clearly enough to prove the case where f is a standard monomial (with coefficient 1). We proceed by induction on $|\mathbf{n}|$ where $f = x_t^{n_t} \cdots x_1^{n_1}$ and $\mathbf{n} = (n_t, \ldots, n_1)$. The result clearly holds for $\mathbf{n} = \mathbf{0}$.

Suppose $\deg[f, x_j] \leqslant_1 \deg f$. Then $[fx_i, x_j] = [f, x_j]\, x_i + f[x_i, x_j]$, so

$$\begin{aligned} \deg[fx_i, x_j] &\leqslant_1 \max\{\deg[f, x_j] + \deg x_i, \deg f + \deg[x_i, x_j]\} \\ &\leqslant_1 \deg f + \deg x_i = \deg fx_i, \end{aligned}$$

since $\deg[f, x_j] \leqslant_1 \deg f$ by the induction hypothesis and $\deg[x_i, x_j] \leqslant_1 \deg x_i$ by complete solvability. This proves the lemma.

By using the ideal generated by all elements of R which occur as coefficients of terms whose degree has norm n (for fixed n) in elements of I whose degree has norm n, Chin has been able to prove results without the hypothesis of complete solvability. The next result is proved in [10, Theorem 2.7]. (The proof given there can be made to yield the semiprime result as well.)

PROPOSITION 4.2. *If L is free as a k-module, then $R \# U(L)$ is prime [resp. semiprime] if and only if R is L-prime [resp. L-semiprime].*

The next result gives yet another characterization of L-[semi]prime ideals and is the exact analogue of a result for strongly group-graded rings. The proof is essentially the same as the proof in that case. (See [1, Lemma 3.3].)

LEMMA 4.3. *If L is free as a k-module, then an ideal I of R is L-prime [resp. L-semiprime] if and only if $I = P \cap R$ for a prime [resp. semiprime] ideal P of $R \# U(L)$.*

Proof. Note that by the Skew PBW Theorem, R does embed in $S = R \# U(L)$.

($\leftarrow$) If A and B are L-invariant ideals of R with $AB \subseteq I$, then AS and BS are ideals of S with $(AS)(BS) \subseteq P$.

($\rightarrow$) Take P to be an ideal of S maximal with respect to $P \cap R = I$. (Such a P exists since $IS \cap R = I$.)

We can now characterize the prime radical of $R \# U(L)$.

PROPOSITION 4.4. *If L is free as a k-module and N_L is the intersection of all L-prime ideals of R, then the prime radical of $R \# U(L)$ is $N_L(R \# U(L)) = (R \# U(L))\, N_L$.*

Proof. Set $S = R \# U(L)$. By Propositions 2.3 and 4.2, $N_L S$ is a semiprime ideal of S. On the other hand, if P is a prime ideal of S, then by Lemma 4.3, $P \cap R$ is an L-prime ideal of R, so $P \cap R \supseteq N_L$. Thus $P \supseteq (P \cap R)\, S \supseteq N_L S$.

Note that if R has the a.c.c. on ideals and N is the prime radical of R, then $N_L = (N : L)$.

5. THE GOLDIE AND JACOBSON CONDITIONS IN SKEW ENVELOPING ALGEBRAS

We now wish to discuss semiprime right Goldie rings, so we need to determine the relation between the Goldie dimensions of R and $R \# U(L)$. (Recall that the *right Goldie dimension* of a ring R is the supremum of the sizes of sets of independent right ideals in R.)

LEMMA 5.1. *Let L be free as a k-module. The right Goldie dimension of R is less than or equal to the right Goldie dimension of $R \# U(L)$, and equality holds if R is a semiprime right Goldie ring.*

Proof. If an independent set of right ideals of R is induced up to $R \# U(L)$, it remains independent (because of the Skew PBW Theorem). This proves the first claim.

Now suppose R is a semiprime right Goldie ring. Our proof below is essentially the same as that of Sigurdsson [29, Lemma 2.4] in the case of a differential operator ring, making use of [15, Lemma 2.1]. (This latter result remains valid with essentially the same proof when we replace $T = R[\theta; \delta]$ by $T = R \# U(L)$.) We provide only an outline.

Since $C = \mathscr{C}_R(0)$ is easily seen to be a right Ore set of regular elements in $R \# U(L)$ and Goldie dimension remains unchanged when we localize at such a set, we may pass to RC^{-1} and $(R \# U(L))C^{-1}$, the latter ring being isomorphic to $RC^{-1} \# U(L)$, where L acts on RC^{-1} in the natural way. (See [3, Theorem 4.4, p. 38].) Thus we may assume R is semisimple Artinian.

Set $S = R \# U(L)$. We know 1 is a sum of orthogonal idempotents $e_1, ..., e_n$ in R such that each $e_i R$ is a simple right ideal. Thus to finish the proof we need only show that if e is an idempotent of R with eR simple, then eS is a uniform right ideal of S. As in [15, Lemma 2.1], we will show that eS is *compressible*, that is, it embeds in each of its nonzero submodules. Since S is Noetherian, this will imply that eS is uniform.

Take any nonzero submodule N of eS and let x in N have smallest degree. Since eR is simple, there is an r in R such that xr has e as its leading coefficient. Since xr also has smallest degree of any element of N, one can show that $r-\operatorname{ann}_S(xr) = r-\operatorname{ann}_S(e) = (1-e)S$. Thus $xrS \cong S/r-\operatorname{ann}_S(xr) \cong eS$. This show eS embeds in N, so eS is indeed compressible.

Note that equality does not hold in general between the Goldie dimensions of R and $R \# U(L)$, even if R is commutative Noetherian and L is Abelian. Let k be a field of characteristic 2, let $R = k[t]/(t^2)$, let L be the Abelian Lie k-algebra with basis x, and let $\delta = \delta_x$ be the derivation of R taking $\overline{p(t)}$ to $\overline{p'(t)}$. Then R is a commutative Artinian ring of Goldie dimension 1, yet $R[x; \delta] = R \# U(L)$ has the nontrivial idempotent xt, so it must have Goldie dimension greater than 1. (It can be shown that $R[x; \delta]$ is isomorphic to the matrix ring $M_2(k[x^2])$ and so has Goldie dimension exactly 2. This example was pointed out to the author by K. R. Goodearl.)

PROPOSITION 5.2. *Let L be free as a k-module.*

(a) *If R has no $\mathbb{Z}$-torsion and $R \# U(L)$ is a prime [resp. semiprime] right Goldie ring, then R is a prime [resp. semiprime] right Goldie ring.*

(b) *If R is a prime [resp. semiprime] right Goldie ring, then $R \# U(L)$ is a prime [resp. semiprime] right Goldie ring.*

Proof. (a) Suppose $R \# U(L)$ is semiprime right Goldie. Since R inherits the a.c.c. on right and left annihilators from $R \# U(L)$, the prime radical N of R is nilpotent by [18, Theorem 1]. By Proposition 2.7, N is L-invariant and by Lemma 4.3, R is L-semiprime, so $N = 0$ and hence R

is semiprime. If $R \# U(L)$ is prime, then R is prime by Theorem 2.8 and Lemma 4.3. Finally, Lemma 5.1 implies R has finite right Goldie dimension.

(b) Suppose R is semiprime right Goldie. By Proposition 4.2, $R \# U(L)$ is semiprime. If $C = \mathscr{C}_R(0)$, then by Goldie's Theorem, RC^{-1} is semisimple Artinian and so $(R \# U(L))\, C^{-1}$, which is isomorphic to $(RC^{-1}) \# U(L)$, is semiprime Noetherian by previous remarks. This implies $R \# U(L)$ is itself semiprime right Goldie.

The next result follows from a result of Jordan [23, Theorem 3.5] on differential operator rings and the fact that skew universal enveloping algebras of completely solvable Lie algebras and of finite-dimensional solvable Lie algebras are iterated differential operator rings. (A *Jacobson ring* is a ring in which every prime ideal is an intersection of primitive ideals.)

PROPOSITION 5.3. *If R is a right Noetherian Jacobson ring and either L is completely solvable or k is a field and L is solvable, then $R \# U(L)$ is a Jacobson ring.*

6. Hypercentral and Simple Skew Enveloping Algebras

We now consider the question of simplicity of $R \# U(L)$ and of differential operator rings. First, we note that R is L-simple if and only if $I \cap R = 0$ for every proper ideal I of $R \# U(L)$. For maximum generality, and because we will need the results in this form later, we will first study iterated differential operator rings S over R, specifying conditions that generalize those of skew enveloping algebras for several classes of solvable Lie algebras.

Thus we will be interested in a ring S for which there is a sequence $R = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_t = S$ such that each $R_i = R_{i-1}[x_i; \delta_i]$ for some derivation δ_i on R_{i-1}. We will also assume that for each i, $\delta_i(R) \subseteq R$, so $\{\delta_1, ..., \delta_t\}$ is a set of derivations on R. To get results we will need some additional restrictions. We will consider three conditions:

$$\text{for } 1 \leqslant j < i \leqslant t,\ \delta_i(x_j) \in R_{j-1}. \tag{1}$$

This will hold, for example, if $S = R \# U(L)$ where L is a finite dimensional nilpotent Lie algebra over a field k. To see this, choose a basis $\{x_1, ..., x_t\}$ for L such that $\{x_1, ..., x_{n_1}\}$ is a basis for the center of L, $\{x_1, ..., x_{n_2}\}$ is a basis for the second higher center of L, and so on. Then $\delta_i(x_j) = [x_i, x_j]$ lies in a lower center of L than x_j, and so is clearly in R_{j-1}.

$$\text{for } 1 \leqslant j < i \leqslant t, \qquad \delta_i(x_j) \in R_j. \tag{2}$$

As remarked before, this will hold if $S = R \# U(L)$ where L is a completely solvable Lie algebra.

There is a sequence $0 = n_0 < n_1 < \cdots < n_k = t$ such that if $1 \leqslant l \leqslant k$, then

$$\text{for} \quad 1 \leqslant j < i \leqslant n_l, \quad \delta_i(x_j) \in R_{n_{l-1}} \quad \text{and} \quad \text{for} \quad 1 \leqslant j \leqslant n_l < i \leqslant t, \quad \delta_i(x_j) \in R_{n_l}. \tag{3}$$

This condition will hold if $S = R \# U(L)$ where L is finite dimensional solvable Lie algebra over a field k. If the kth derived subalgebra $L^{(k)}$ is 0, but $L^{(k-1)}$ is not 0, we can see that (3) holds by choosing a basis $\{x_1, \dots, x_t\}$ for L such that $\{x_1, \dots, x_{n_m}\}$ is a basis for $L^{(k-m)}$.

Note that $(1) \to (2) \to (3)$. Note also that if $i > j$ (and $j = n_l$ for some l in case (3)), then R_i is an iterated differential operator ring over R_j satisfying the same condition as S does over R.

In any iterated differential operator ring, one can still define standard monomials, degree, leading coefficient, and so on (although degree may not have the same nice properties as in skew enveloping algebras, even if condition (1) holds). We will make use of these concepts in the following results.

We remarked at the beginning of Section 3 that $R \# U(L)$ is right Noetherian whenever R is, provided that L is finitely generated; this can be shown using general results on filtrations and gradings. If S is an iterated differential operator ring over R, one can show that S is right Noetherian whenever R is by using a noncommutative version of the Hilbert Basis Theorem. We state a stronger result for two-sided ideals which is a variant of the Hilbert Basis Theorem and applies to skew enveloping algebras of finite dimensional nilpotent Lie algebras.

PROPOSITION 6.1. *If S is an iterated differential operator ring over R with each $\delta_i(R) \subseteq R$ and if condition* (1) *above holds, then R has the* a.c.c. *on $\{\delta_1, \dots, \delta_t\}$-invariant ideals if and only if S has the* a.c.c. *on ideals.*

Proof. Sufficiency is clear. To prove necessity, it suffices by induction on t to show that $R_1 = R[x_1; \delta_1]$ has the a.c.c. on $\{\delta_2, \dots, \delta_t\}$-invariant ideals, since S satisfies condition (1) as an iterated differential operator ring over R_1.

For an ideal I of R_1 and a natural number n, let

$$\operatorname{ld}_n(I) = \left\{ r \in R \mid x_1^n r + \sum_{i=0}^{n-1} x_1^i r_i \in I \text{ for some } r_i \in R \right\}.$$

If I is a $\{\delta_2, \dots, \delta_t\}$-invariant ideal of R_1, it is not hard to check using condition (1) that $\operatorname{ld}_n(I)$ is a $\{\delta_1, \dots, \delta_t\}$-invariant ideal of R. (For any ideal I, it is a δ_1-invariant ideal of R.) It is also easy to check that if J and I are ideals of R_1 and $J \subseteq I$ and $\mathrm{ld}_n(J) = \mathrm{ld}_n(I)$ for all $n \in \mathbb{N}$, then $I = J$.

Now suppose $I_0 \subseteq I_1 \subseteq \cdots$ is an ascending chain of $\{\delta_2, ..., \delta_t\}$-invariant ideals of R_1. Then for each n we get a chain $\mathrm{ld}_n(I_0) \subseteq \mathrm{ld}_n(I_1) \subseteq \cdots$, of $\{\delta_1, ..., \delta_t\}$-invariant ideals of R. For each k we also have a chain $\mathrm{ld}_0(I_k) \subseteq \mathrm{ld}_1(I_k) \subseteq \cdots$. We can combine these chains to get a new chain $\mathrm{ld}_0(I_0) \subseteq \mathrm{ld}_1(I_1) \subseteq \cdots$. Using the a.c.c. in R on this chain and noting the inclusions in the original sets of chains, it is easy to see that there is a l such that $\mathrm{ld}_n(I_m) = \mathrm{ld}_l(I_l)$ for n, $m \geqslant l$.

One can now apply the a.c.c. in R to the chains $\mathrm{ld}_j(I_0) \subseteq \mathrm{ld}_j(I_1) \subseteq \cdots$, for $0 \leqslant j < l$, and get an $l' \geqslant l$ such that $\mathrm{ld}_j(I_n) = \mathrm{ld}_j(I_m)$ for all n, $m \geqslant l'$ and all j. By the remarks above, this implies that $I_n = I_m$ for all n, $m \geqslant l'$.

If Δ is a set of derivations on R, we say R is *Δ-hypercentral* if for any two Δ-invariant ideals I_1 and I_2 of R with $I_1 \subset I_2$, there is an element $r \in I_2 \backslash I_1$ such that $r + I_1$ is central in R/I_1 and $\delta(r) - r \in I_1$ for each $\delta \in \Delta$. The next results are the analogues of our hypercentrality results for strongly group-graded rings. The following result can be regarded as a generalization of the fact that if R is a δ-simple ring, every ideal of $R[x; \delta]$ is generated by a central element, namely a monic element of least degree.

PROPOSITION 6.2. *Suppose S is an iterated differential operator ring over R with each $\delta_i(R) \subseteq R$ and condition* (1) *holds. If R is a $\{\delta_1, ..., \delta_t\}$-hypercentral ring, then S is a hypercentral ring.*

Proof. It is clearly enough to show $R_1 = R[x_1; \delta_1]$ is a $\{\delta_2, ..., \delta_t\}$-hypercentral ring if R is a $\{\delta_1, ..., \delta_t\}$-hypercentral ring. Let I_1 and I_2 be $\{\delta_2, ..., \delta_t\}$-invariant ideals of R_1. Suppose $f \in I_2 \backslash I_1$ has least degree in x_1, say f has degree m. For an ideal I of R_1, let

$$\mathrm{ld}(I) = \left\{ r \in R \mid x_1^m r + \sum_{i=0}^{m-1} x_1^i r_i \in I \text{ for some } r_0, ..., r_{m-1} \in R \right\}.$$

Clearly $\mathrm{ld}(I)$ is an ideal of R and $\mathrm{ld}(I_1) \subseteq \mathrm{ld}(I_2)$. By choice of m, it is easy to see $\mathrm{ld}(I_1) \subset \mathrm{ld}(I_2)$. We claim the $\mathrm{ld}(I_k)$ are $\{\delta_1, ..., \delta_t\}$-invariant ideals of R. To see this, suppose $g = \sum_{i=0}^{m} x_1^i r_i \in I_k$, so $r = r_m \in \mathrm{ld}(I_k)$. Then for any j with $1 \leqslant j \leqslant t$, we have $\delta_j(g) \in I_k$. (Here $\delta_1(g) = x_1 g - g x_1$, which is in I_k since it is an ideal of R_1.) By condition (1), each $\delta_j(x_1) \in R$, so each $\delta_j(x_1^i)$ has degree at most $i - 1$. This implies that the only term in $\delta_j(g)$ containing x_1^m is $x_1^m \delta_j(r_m)$. Thus $\delta_j(r) \in \mathrm{ld}(I_k)$, proving $\mathrm{ld}(I_k)$ is δ_j-invariant.

Thus by the hypercentral property, there is an element $g = x_1^m r + \sum_{i=0}^{m-1} x_1^i r_i$ of $I_2 \backslash I_1$ such that $r \in \mathrm{ld}(I_2) \backslash \mathrm{ld}(I_1)$ and $r + \mathrm{ld}(I_1)$ is central in $R/\mathrm{ld}(I_1)$ and each $\delta_j(r) - r \in \mathrm{ld}(I_1)$. It is easy to see that g must be central mod I_1 and $\delta_j(g) - g \in I_1$ for $2 \leqslant j \leqslant n$. (The proof uses minimality of m and is very similar to the proof of [1, Proposition 5.1].)

Proposition 6.2 enables us to get criteria for simplicity of skew enveloping algebras corresponding to those we gave for crossed products in [1, Theorem 5.6]. Recall that for a ring R and an element λ of R, we denote by ad λ the inner derivation of L defined by $\operatorname{ad}\lambda(r) = \lambda r - r\lambda$. Denote by R^L the set of L-*constants* of R, that is the set of $r \in R$ such that $\delta_l(r) = 0$ for all $l \in L$, and denote by $Z(L)$ the center of L.

PROPOSITION 6.3. *Suppose k is a field and L is a finite dimensional nilpotent Lie k-algebra with basis $\{x_1,..., x_t\}$.*

(a) *The ring $R \# U(L)$ is a simple ring if and only if R is L-simple and the center of $R \# U(L)$ is a field.*

(b) *Suppose R is L-simple and k has characteristic 0. If there do not exist $\mu_1,..., \mu_t$ in $R^{Z(L)} \cap \operatorname{cen} R$, not all zero, and $\lambda \in R^{Z(L)}$ such that $\mu_1 \delta_{x_1} + \cdots + \mu_t \delta_{x_t} = \operatorname{ad}\lambda$, then $R \# U(L)$ is a simple ring. In case these conditions are satisfied,* $\operatorname{cen} R \# U(L) = R^L \cap \operatorname{cen} R$.

Proof. The equivalence in (a) is clear from Proposition 6.2. To prove (b) it suffices to show if the stated hypotheses hold, the center of $R \# U(L)$ is the field $R^L \cap \operatorname{cen} R$. Since it is clear that $R^L \cap \operatorname{cen} R = R \cap \operatorname{cen} R \# U(L)$, it is enough to show that all elements of the center of $R \# U(L)$ have degree $\mathbf{0}$.

Suppose $f \in \operatorname{cen}(R \# U(L))$ and $f \notin R$, so $m = |\deg f| > 0$. We can write

$$f = f_1 r_1 + \cdots + f_a r_a + g_1 s_1 + \cdots + g_b s_b + g,$$

where $r_1,..., r_a \in R$ are nonzero, $s_1,..., s_b \in R$, and the elements f_i and g_j are distinct standard monomials with $|\deg f_i| = m$ for each i, $|\deg g_j| = m - 1$ for each j, and $|\deg g| \leqslant m - 2$. Clearly for $l \in Z(L)$.

$$0 = [f, l] = -\sum f_i \delta_l(r_i) + -\sum g_j \delta_l(s_j) + [g, l],$$

so each $\delta_l(r_i) = 0$ and each $\delta_l(s_j) = 0$, i.e., $r_1,..., r_a, s_1,..., s_b \in R^{Z(L)}$. If $r \in R$, then

$$\begin{aligned} 0 = [f, r] &= \sum [f_i, r]\, r_i + \sum [g_j, r]\, s_j + [g, r] \\ &\quad + \sum f_i(r_i r - r r_i) + \sum g_j(s_j r - r s_j). \end{aligned} \tag{4}$$

Since the elements f_i and g_j are monic, $|\deg[f_i, r]| < |\deg f_i| = m$ and $|\deg[g_j, r]| < |\deg g_j| = m - 1$; also $|\deg[g, r]| < |\deg g| \leqslant m - 2$. Thus the only terms in the right-hand side of expression (4) for $[f, r]$ whose degrees could have norm m are the terms $f_i(r_i r - r r_i)$. Since the f_i are independent over R and $[f, r] = 0$, this means that each $r_i r = r r_i$, so $r_1,..., r_a \in \operatorname{cen} R$.

Let us consider what terms can occur in the right hand side of (4) with degrees of norm $m-1$. The only possibilities are the terms in $[f_i, r]\, r_i$ and the terms $g_j(s_j r - rs_j)$. Suppose $f_i = x_t^{n(t,i)} \cdots x_1^{n(1,i)}$; then

$$[f_i, r] = -\sum_{h=1}^{t} x_t^{n(t,i)} \cdots x_h^{n(h,i)-1} \cdots x_1^{n(1,i)} n(h, i)\, \delta_{x_h}(r)$$
$$+ \text{ terms of lower norm.} \tag{5}$$

Since $f \notin R$, some $n_{h,i} \neq 0$, so for simplicity suppose $n_{l,1} \neq 0$. Then

$$[f_1, r]\, r_1 = -x_t^{n(t,1)} \cdots x_l^{n(l,1)-1} \cdots x_1^{n(1,1)} n_{l,1}\, \delta_{x_l}(r)\, r_1$$
$$+ \text{ terms of different degree.} \tag{6}$$

The only other $[f_i, r]\, r_i$ which could contain such a term is one where $f_{i(h)}$ contains $x_h^{n(h,1)+1}$ along with the same powers of x_i for $i \neq h$ as in (6). For such a monomial, $[f_{i(h)}, r]\, r_{i(h)}$ contributes a term of the same degree as in (6) with coefficient $-(n(h,1)+1)\, \delta_{x_h}(r)\, r_{i(h)}$ plus terms of other degrees. If we add up all such terms we get the term

$$-x_t^{n(t,1)} \cdots x_l^{n(l,1)-1} \cdots x_1^{n(1,1)} \left(\sum_{h=1}^{t} \delta_{x_h}(r)\, \mu_h \right), \tag{7}$$

where $\mu_h = 0$ if no $i(h)$ exists, $\mu_l = n(l,1)\, r_1$, and otherwise $\mu_h = (n(h,1)+1)\, r_{i(h)}$. Note that by our work at the start of the proof, each $\mu_h \in R^{Z(L)} \cap \operatorname{cen} R$, and $\mu_l \neq 0$ since k has characteristic 0.

There can be at most one term of the form $g_j(s_j r - rs_j)$ which yields a monomial term of the form $x_t^{n(t,1)} \cdots x_l^{n(l,1)-1} \cdots x_1^{n(1,1)} s$, and for such a j, the coefficient s is $s_j r - rs_j$. For such a j, set $\lambda = s_j$; if no such j exists, set $\lambda = 0$. We thus see that $[f, r]$ contains the term

$$x_t^{n(t,1)} \cdots x_l^{n(l,1)-1} \cdots x_1^{n(1,1)} \left(-\sum_{h=1}^{t} \mu_h\, \delta_{x_h}(r) + \operatorname{ad} \lambda(r) \right)$$

so since $[f, r] = 0$ for every $r \in R$, we must have $\operatorname{ad} \lambda = \sum_{h=1}^{t} \mu_h\, \delta_{x_h}$. This violates (b), so in fact $\operatorname{cen}(R \# U(L)) \subseteq R$ and hence is a field.

An easy corollary of this is a result of Hauger [17]: if L is an Abelian Lie k-algebra which is a free k-module with basis $\{x_1, \ldots, x_t\}$ and R has no $\mathbb{Z}$-torsion, then $R \# U(L)$ is a simple ring if and only if R is L-simple and there do not exist $\mu_1, \ldots, \mu_t \in R^L \cap \operatorname{cen} R$, not all zero, and $\lambda \in R^L$ such that $\mu_1 \delta_{x_1} + \cdots + \mu_t \delta_{x_t} = \operatorname{ad} \lambda$. For more results in this direction, including results for rings with $\mathbb{Z}$-torsion, see [17] and [16, Sect. 2].

7. Localization and the Second Layer Condition in Skew Enveloping Algebras

We now investigate localization and the second layer condition for skew enveloping algebras and iterated differential operator rings. First, we will summarize some of the results we will need. For a more detailed description of the theory of localization we will apply, see the memoir of Jategaonkar [22] or the survey of Brown [7]. (Also see the introduction to Section 6 in [1] where we prove some of the following assertions in detail.)

If Q and P are prime ideals of a Noetherian ring R, we say Q is *linked* to P, denoted $Q \rightsquigarrow P$, if there is an ideal A of R with $QP \subset A \subseteq Q \cap P$ such that $(Q \cap P)/A$ is torsionfree as a right R/P- and left R/Q-module. We define the *clique* of P to be smallest set of prime ideals of R containing P and closed under links (to the right or the left). Following Goldie's Theorem, to localize at the prime ideal P, we would like to invert the elements of $\mathscr{C}(P) = \{r \in R \mid r + P \text{ is a regular element in } R/P\}$, so we would like $\mathscr{C}(P)$ to be an Ore set in R. However, it is known (see [22, Theorem 5.4.5; 7, Lemma 1.1] that any Ore set C contained in $\mathscr{C}(P)$ is contained in $\mathscr{C}(X) = \bigcap_{Q \in X} \mathscr{C}(Q)$, where X is the clique of P, so the best we can hope for is that $\mathscr{C}(X)$ is an Ore set. We define the clique X to be *classically localizable* if $\mathscr{C}(X)$ is an Ore set and some technical conditions hold—see [22, Sect. 7.1] or [7, Definition 3.1]. The question thus becomes: when is X classically localizable?

In [20], Jategaonkar introduced a condition which enables one to develop a reasonable theory of localization in Noetherian rings, and in [22], he has worked out a great deal of this theory. The condition, now called the *second layer condition*, is defined as follows. If R is a right Noetherian ring, M is a uniform right R-module, and P is a prime ideal of R, we say M is *P-tame* if M contains a copy of a uniform right ideal of R/P. We say a right Noetherian ring R satisfies the *right second layer condition* if for any finitely generated tame uniform right R-module M such that $\operatorname{ann} M$ is prime, the annihilator of every nonzero submodule of M is $\operatorname{ann} M$. We say R satisfies the *strong right second layer condition* if the above condition holds for any finitely generated uniform right R-module M such that $\operatorname{ann} M$ is prime. (We no longer require M to be tame.)

Suppose R is a Noetherian ring satisfying the second layer condition. Jategaonkar [22, Theorem 7.2.5] has shown that any finite clique in R is classically localizable. If a prime ideal P has the AR property (see below), then it is known that P cannot be linked to any prime ideal different from itself (see the introduction to Sect. 6 in [1]), whence it follows that P is classically localizable. (For a prime ideal P, classical localizability is equivalent to $\mathscr{C}(P)$ being an Ore set in R and the Jacobson radical of the localization R_P having the AR property.) If the clique X is infinite, open questions remain. It has been shown by Warfield and Stafford (see [22, Theorem 7.2.15; 7, Theorem 3.7]) that a clique X is classically localizable provided that R is an algebra over an uncountable field and either R is fully bounded or there is a finite bound on the Goldie dimensions of the rings R/P as P runs through X.

In the proof of the second layer condition, we will make use of the Artin–Rees property. (For a more detailed study of the results mentioned here, see [1, Sect. 6].) The Artin–Rees property is discussed in [24; 9, Chap. 11].

An ideal I of R is said to be *right* AR or to have the *right* AR *property* if for any right ideal K of R there is a positive integer m such that $K \cap I^m \subseteq KI$. We need to make use of another form of the AR property which implies the usual AR property (by a non-commutative version of the Artin–Rees lemma). We will say an ideal I has the *very strong right* AR *property* if the so-called *Rees ring of I over R*, that is the subring $R^*(I) = R + xI + x^2I^2 + \cdots$, of the polynomial ring $R[x]$, is right Noetherian. (For another formulation and a proof that this property implies the usual AR property, see [1, Sect. 6].) It is easy to check that if $\pi\colon R \to \bar{R}$ is a surjective ring homomorphism and I has either form of the AR property in R, then $\pi(I)$ has the same property in $\bar{R}$.

McConnell [24, Corollary 10] has shown by iterated use of the Hilbert Basis Theorem that the Rees ring $R^*(I)$ is right Noetherian if the ideal I is generated by a commuting set of normalizing elements of the right Noetherian ring R; thus such an ideal has the very strong right AR property in a right Noetherian ring R. (An element a of R is called a *normalizing* or *normal* element if $aR = Ra$.) We will make use of this fact throughout this section, especially for ideals generated by central elements.

Suppose Q and P are prime ideals of R and $Q \subset P$. We will call (Q, P) an *undesirable pair of prime ideals* if there exists a finitely generated uniform right R-module M with ann $M = Q$, which contains a copy of a uniform right ideal of R/P. Thus R satisfies the right second layer condition precisely when it has no undesirable pair of prime ideals. The utility of the AR property for us is that if Q and P are prime ideals of R for which there exists an ideal I with $Q \subset I \subseteq P$ and with I/Q a right AR ideal of R/Q, then (Q, P) cannot be an undesirable pair. The proof of this remark is essentially the same as the proof of [21, Proposition 4.1; 1, Proposition 6.5]. (If for every pair of prime ideals Q, P with $Q \subset P$ such an ideal I exists, we say R is a *poly*-AR or AR *separated* ring. Thus a Noetherian AR separated ring satisfies the second layer condition.)

If L is a finite dimensional solvable Lie algebra over an algebraically closed field k of characteristic zero, then McConnell [24, Theorem 3] has proved that ideals in $U(L)$ have normalizing sets of generators; moreover, McConnell's proof remains valid over an arbitrary field k of characteristic zero if L is completely solvable. It follows from [24, Lemma 8] that $U(L)$ is an AR separated ring. Jategaonkar [21, Theorem 4.2] deduces from this that $U(L)$ satisfies the second layer condition for finite dimensional solvable L and any field k of characteristic zero. (Localization in $U(L)$ was also studied independently by Brown in [4, 5, 6].) If L is a finite dimensional Lie algebra over a field of non-zero characteristic, then [19, Lemma VI.5, p. 204; Lemma V.4, p. 189] shows that $U(L)$ is a finitely generated module over its center. Thus $U(L)$ is a Noetherian p.i. ring and so satisfies the second layer condition. (One can show directly that all cliques are finite and localizable in a ring which is finite as a module over a commutative Noetherian ring, as in [26, Theorem 7].)

We would like to show that if k is a field, R is a commutative Noetherian k-algebra, and L is a finite dimensional solvable Lie k-algebra, then $R \# U(L)$ satisfies the second layer condition. However, we have been forced to assume $R \supseteq \mathbb{Q}$, unless L is actually nilpotent. Some of the following results are accordingly divided into two cases. As before, we will state our results in terms of iterated differential operator rings to achieve maximum generality and to enable us to proceed inductively.

LEMMA 7.1. *Suppose R is a right Noetherian ring and S is an iterated differential operator ring over R with each $\delta_i(R) \subseteq R$. If I is a $\{\delta_1, \ldots, \delta_t\}$-invariant ideal of R which is very strongly right* AR, *then IS is a very strongly right* AR *ideal of S.*

Proof. It is easy to see that $IS = SI$ and that IS is an ideal of S. Let $R^*(I)$ be the subring $R + xI + x^2I^2 + \cdots$, of $R[x]$. We can define $S[x]$ to be an iterated differential operator ring over $R[x]$ by simply declaring x to be a δ_i-constant for each i. Since I is invariant under each δ_i, we can also define an iterated differential operator subring $S^*(I)$ of $S[x]$ by setting $S^*(I) = R^*(I)[x_1; \delta_1] \cdots [x_t; \delta_t]$. Since $R^*(I)$ is right Noetherian, the Hilbert Basis Theorem implies $S^*(I)$ is right Noetherian. Because x commutes with everything, it is clear that $S^*(I) = S + xIS + x^2(IS)^2 + \cdots$, so IS is a very strongly right AR ideal of S.

LEMMA 7.2. *Suppose R is right Noetherian and S is an iterated differential operator ring over R with each $\delta_i(R) \subseteq R$. Then there is no undesirable pair (Q, P) of prime ideals in S with $Q \cap R = P \cap R$ if either condition* (1) *of Section* 6 *holds or R is a $\mathbb{Q}$-algebra and condition* (3) *of Section* 6 *holds.*

Proof. Since each $\delta_i(R) \subseteq R$, it follows that each $\delta_i(Q \cap R) \subseteq Q \cap R$. Because of this containment, we know that $(Q \cap R) R_i \cap R_j = (Q \cap R) R_j$ if $j < i$, and hence we may factor $(Q \cap R) R_i$ out of each R_i. Thus we may pass to $R/(Q \cap R)$ and so assume $Q \cap R = P \cap R = 0$. As in the proof of

Lemma 4.3, this implies that R is $\{\delta_1,..., \delta_t\}$-prime. If $C = \mathscr{C}_R(0)$, then by Proposition 2.12, C is a right Ore set in R and RC^{-1} is a right Artinian ring. It is easy to check that C is a right Ore set of regular elements in S. (See [3, Theorem 4.4, p. 38].) If we extend the derivations δ_i to derivations $\tilde{\delta}_i$ on RC^{-1} in the natural way, we can easily see that SC^{-1} is an iterated differential operator ring over RC^{-1} satisfying the same condition as S does over R. Proposition 2.12 also implies that RC^{-1} is $\{\tilde{\delta}_1,..., \tilde{\delta}_t\}$-simple. Furthermore, (QC^{-1}, PC^{-1}) is still an undesirable pair of prime ideals in SC^{-1} by [1, Lemma 6.6].

Thus we may assume R is right Artinian and $\{\delta_1,..., \delta_t\}$-simple. First, suppose that condition (1) is satisfied. Then by Proposition 6.2, S is a hypercentral ring, so S is an AR separated ring (actually every ideal in R has the AR property by [27, Sect. 2.7]). This implies no undesirable pair of prime ideals can exist in S.

Suppose now that $R \supseteq \mathbb{Q}$ and that condition (3) holds. By Corollary 2.10, R is a simple Artinian ring. We know that there is a sequence $R = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = S$ such that each

$$S_{l+1} = S_l[x_{n_l+1}; \delta_{n_l+1}] \cdots [x_{n_{l+1}}; \delta_{n_{l+1}}],$$

and for $n_l + 1 \leqslant i, j \leqslant n_{l+1}$, we have $x_i x_j - x_j x_i \in S_l$ and for any $i > n_l$, we have $\delta_i(S_l) \subseteq S_l$. By induction on k, we may suppose $Q \cap S_1 \subset P \cap S_1$. Since $R = S_0$ is simple and S_1 satisfies condition (1) as an iterated differential operator ring over S_0, we may apply Proposition 6.2 and conclude that $(P \cap S_1)/(Q \cap S_1)$ contains a non-zero central element of $S/(Q \cap S_1)$. Since for $i > n_1$, we have $\delta_i(S_1) \subseteq S_1$, the ideals $Q \cap S_1$ and $P \cap S_1$ are $\{\delta_{n_1+1},..., \delta_t\}$-invariant. Thus we may pass to $S_1/(Q \cap S_1)$ and assume $Q \cap S_1 = 0$.

Since $P \cap S_1$ is $\{\delta_{n_1+1},..., \delta_t\}$-invariant, the remark before Proposition 2.1 implies the ideal I generated by the central elements of $P \cap S_1$ is a nonzero $\{\delta_{n_1+1},..., \delta_t\}$-invariant ideal of S_1 with the very strong right AR property. Thus by Lemma 7.1, IS is a very strongly right AR ideal of S contained in P but not in Q. This implies that (Q, P) cannot be an undesirable pair of prime ideals, since $Q \subset Q + IS \subseteq P$ and $(Q + IS)/Q$ is a homomorphic image of IS in S/Q and hence has the AR property. This proves the lemma.

THEOREM 7.3. *Suppose R is a right Noetherian ring and S is an iterated differential opertor ring over R with each $\delta_i(R) \subseteq R$. If for any two $\{\delta_1,..., \delta_t\}$-prime ideals I and J of R with $I \subset J$, there is a $\{\delta_1,..., \delta_t\}$-invariant ideal J' of R with $I \subset J' \subseteq J$, such that J'/I is a very strongly right* AR *ideal in R/I, then S satisfies the second layer condition if either condition* (1) *in Section* 6 *is satisfied or $R \supseteq \mathbb{Q}$ and condition* (3) *in Section* 6 *is satisfied.*

Proof. Assume (Q, P) is an undesirable pair of prime ideals in S. If $Q \cap R = P \cap R$, we are done by Lemma 7.2. If $Q \cap R \subset P \cap R$, then there is a $\{\delta_1, ..., \delta_t\}$-invariant ideal I of R with $Q \cap R \subset I \subseteq P \cap R$ such that $I/Q \cap R$ is very strongly right AR in R. The ideal $K = IS + Q$ is contained in P but strictly contains Q, and K/Q has the AR property in S/Q since $IS/(Q \cap R) S$ has the AR property in $S/(Q \cap R) S$ by Lemma 7.1. Thus (Q, P) cannot be an undesirable pair of prime ideals.

COROLLARY 7.4. *Suppose k is a field of characteristic zero and L is a finit dimensional solvable Lie k-algebra. Then $R \# U(L)$ satisfies the right second layer condition if either R is a right Artinian ring, R is a simple right Noetherian ring, R is a right Noetherian* p.i. *ring, or R is a principal ideal ring.*

Note that if R is a commutative Noetherian ring, it satisfies the hypothesis of this result.

Proof. The proof is just like that of [1, Corollary 7.4]: in each case we verify the AR condition in the hypotheses of Theorem 7.3. In the p.i. case we use the well-known fact that every non-zero ideal in a prime p.i. ring contains a central element (see [28, Theorem 1.6.27]). In the case of a principal ideal ring R, we use the fact that every ideal of R is generated by a normalizing element and hence has the very strong AR property (see [1, Corollary 6.4]). In the other two cases, we note that there cannot be a strict inclusion of L-prime ideals.

COROLLARY 7.5. *Suppose k is a field and L is a finite dimensional nilpotent Lie k-algebra. Then $R \# U(L)$ satisfies the right second layer condition if either R is a right Artinian ring, R is an L-simple right Noetherian ring, R is a commutative Noetherian ring, or R is a principal ideal ring.*

Proof. Just like the proof of Corollary 7.4.

The same open questions mentioned at the end of [1] apply to the results of this paper, namely (i) do the rings $R \# U(L)$ actually satisfy the strong second layer condition and (ii) to prove $R \# U(L)$ satisfies the second layer condition for L solvable, is it enough to assume that R is a Noetherian ring satisfying the second layer condition?

Combining the localization theory with Corollary 7.4 and a result of Sigurdsson, we get the following result.

PROPOSITION 7.6. *If k is an uncountable field of characteristic zero, R is a Noetherian* p.i. *algebra over k, and L is a finite dimensional solvable Lie k-algebra, then all cliques in $R \# U(L)$ are classically localizable.*

Proof. Since $R \# U(L)$ is an iterated differential operator ring over R, [29, Corollary 2.5] implies that if there is a bound on the Goldie dimension of prime factor rings of $R[x]$, there is a bound on the Goldie dimension of prime factor rings of $R \# U(L)$. By the remarks in the introduction to this section, this will prove every clique in $R \# U(L)$ is localizable.

Since R is a p.i. algebra, it satisfies a multilinear identity of degree n with coefficients in k, for some n. Thus every prime factor ring of $R[x]$ satisfies the same identity, and so the Goldie quotient rings of all prime factor rings of $R[x]$ satisfy the same identity, since they are central localizations (see [28, Proposition 1.7.8 and Theorem 1.7.9, pp. 52, 53]). Kaplanksy's Theorem (see [28, Theorem 1.5.16, p. 36]) implies that the localizations and hence the original factor rings have Goldie dimension at most $n/2$.

Sigurdsson [30] has shown that for a commutative Noetherian ring R containing $\mathbb{Q}$, all cliques in $R[x; \delta]$ consist of a single prime ideal or a countably infinite set of prime ideals and that all cliques are localizable. Brown has described cliques in solvable Lie algebras over $\mathbb{C}$ in [5, 6]: again they consist of either a single prime ideal or are countably infinite. In [31], Sigurdsson considers solvable Lie algebras over other fields of characteristic zero. If L is a nonsolvable Lie algebra in characteristic zero, there are cliques in $U(L)$ which are not localizable. (See the comments in [7] after Remark 2.3.)

Proposition 7.6 shows that cliques are localizable in many cases. We now discuss when single prime ideals are classically localizable. When L is a finite dimensional nilpotent Lie algebra it is well known that all prime ideals in $U(L)$ are classically localizable, since all ideals have centralizing sets of generators. (See [24, Corollary 7 and remarks following].) Our first result generalizes this.

PROPOSITION 7.7. *If L is a finite dimensional nilpotent Lie algebra over the field k and R is an L-hypercentral Noetherian ring, then all prime ideals in $R \# U(L)$ are classically localizable.*

Proof. By Proposition 6.2, all ideals of R have centralizing sets of generators and hence have the AR property. Thus $R \# U(L)$ is a Noetherian ring satisfying the second layer condition and with all primes classically localizable.

Even if L is Abelian and R is a commutative Noetherian ring there may be prime ideals in $R \# U(L)$ that fail to be localizable. One example is the differential operator ring $R[x; \delta] = R \# U(L)$ where $R = k[t]$ for k a field of characteristic zero and $\delta(p(t)) = tp'(t)$. (Here $L = kt$. The ring $R[x; \delta]$ is actually the universal enveloping algebra of a two-dimensional solvable Lie k-algebra.) In this ring prime ideals are usually linked to other primes and hence are rarely localizable. (See [30; 7, Example 1.3].)

Sometimes specific primes in $R \# U(L)$ can be shown to be localizable, as in the next two results.

PROPOSITION 7.8. *If k is a field, L is a finite dimensional nilpotent Lie algebra, and R is a Noetherian ring, then any prime ideal P in $R \# U(L)$ for which $P \cap R = 0$ is classically localizable.*

Proof. Let C be the set of regular elements in R. By Proposition 2.12, C is an Ore set in R and if we let L act on RC^{-1} in the obvious way, RC^{-1} is L-simple. It is not hard to see that C is an Ore set in $R \# U(L)$ and $(R \# U(L))\, C^{-1} = RC^{-1} \# U(L)$. It is well-known that since $P \cap C = \phi$, we have $C \subseteq \mathscr{C}_{R \# U(L)}(P)$ and that PC^{-1} is a prime ideal of $RC^{-1} \# U(L)$. (See [3, 2.10, p. 21].) Since RC^{-1} is L-simple, the last proposition implies that PC^{-1} is classically localizable in $RC^{-1} \# U(L)$. Since $C \subseteq \mathscr{C}(P)$, this implies P itself is localizable.

The following result shows that induced primes are often localizable.

PROPOSITION 7.9. *Suppose k is a field, R is a commutative Noetherian ring, and either L is nilpotent or k has characteristic zero and L is solvable. Then if I is an L-prime ideal of R, $I \# U(L)$ is a classically localizable prime ideal of $R \# U(L)$.*

Proof. By Lemma 7.1, $P = I \# U(L)$ has the AR property, and by Proposition 4.2, P is prime. Since $R \# U(L)$ satisfies the second layer condition, the general localization theory implies P is classically localizable, as noted in the introduction to this section.

We can apply a result of Brown and Warfield to get the following application of localization. (In [14], it is proved that the Krull dimension of R is bounded by the global dimension of R for any Noetherian p.i. algebra R of finite global dimension. In general it is an open question whether the existence of an uncountable central subfield is necessary for the localizability of cliques and hence whether uncountability is necessary in the next result.) Recall that the *classical Krull dimension* of a ring is the supremum of the lengths of strictly increasing chains of prime ideals in the ring.

PROPOSITION 7.10. *If R is a Noetherian* p.i. *algebra over an uncountable field k of a characteristic* 0 *and L is a finite dimensional solvable Lie k-algebra, then the classical Krull dimension of $R \# U(L)$ is bounded by the global dimension of $R \# U(L)$, if the latter is finite. (The global dimension of $R \# U(L)$ is finite if the global dimension of R is.)*

Proof. Since $R \# U(L)$ is an iterated differential operator ring over R, it has finite global dimension if R does. (See, e.g., [13, Proposition 3].) Since all cliques in $R \# U(L)$ are classically localizable by Proposition 7.6, the global dimension of $R \# U(L)$ bounds the classical Krull dimension of $R \# U(L)$ by [8, Theorem 8].

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